Activity 1
Folding Equilateral Triangles in a Square

For courses: geometry, precalculus, calculus (optimization), modeling

Summary Students are asked to find a way to fold an equilateral triangle from a square piece of paper. Then the challenge of finding the largest possible equilateral triangle that can be folded from a square is given. Of course, students need to prove that their conjectured triangle is the largest possible.

Content The geometry component of this problem only requires the ability to work with 30°-60°-90° triangles. However, more creative geometrical insights can lead to more elegant solutions.

For a calculus class, this problem could actually be posed without any mention of origami: What is the largest equilateral triangle that can be inscribed in a square? But knowing that paper folders actually use this knowledge can provide extra motivation. This is a challenging modeling problem that can be completely done without resorting to derivatives, provided the students set up the model carefully, know trigonometry solidly, and do a proper graphical analysis. As an optimization problem, it breaks away from the mold that is typically encountered in calculus textbooks, thus forcing students to apply their knowledge to a brand-new, real-life situation.

Handouts Three optional handouts are provided:

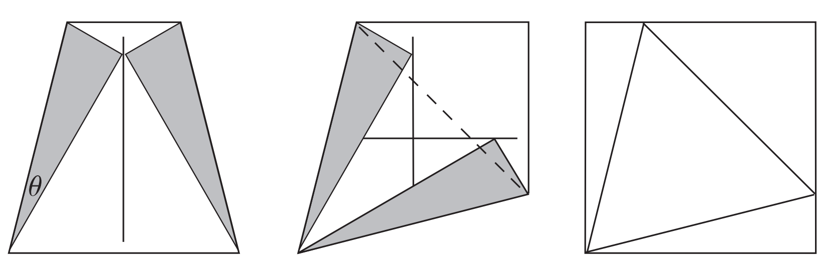
(1) Introduces the general problem of folding an equilateral triangle inside a square.

(2) Provides a few guided steps in setting up the optimization model.

(3) Leads students step-by-step through the optimization model.

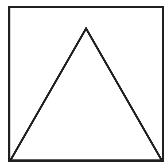
Time commitment Handout 1 will require about 40 minutes of class time, including student exploration and presentation of their triangle-folding methods to the rest of the class.

Handout 2 or 3, if done in class, could take 50–60 minutes total, depending on how quick your students are at making mathematical models.



Handout: How to Fold an Equilateral Triangle?

The goal of this activity is to fold an equilateral triangle from a square piece of paper.



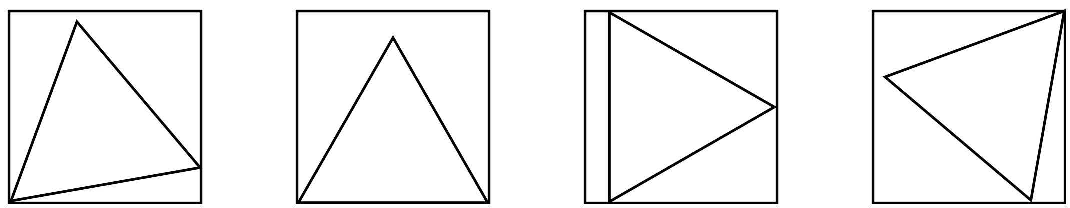
Question 1: First fold your square to produce a 30°-60°-90° triangle inside it. Hint: you want your folds to make the hypothenuse twice as long as one of the sides. Keep trying! Explain why your method works in the space below.

Question 2: Now use what you did in Question 1 to fold an equilateral triangle inside a square.

Follow-up: If the side length of your original square is 1, what is the length of a side of your equilateral triangle? Would it be possible to make the triangle's side length bigger?

Handout
What's the Biggest Equilateral Triangle in a Square?

If we are going to turn a square piece of paper into an equilateral triangle, we'd like to make the **biggest possible** triangle. In this activity your task is to make a mathematical model to find the equilateral triangle with the **maximum area** that we can fit inside a square. Follow the steps below to help set up the model.



Question 1: If such a triangle is maximal, then can we assume that one of its corners will coincide with a corner of the square? Why?

Question 2: Assuming Question 1, draw a picture of what your triangle-in-the-square might look like, where the "common corner" of the triangle and square is in the lower left. Now you'll need to create your model by introducing some variables. What might they be? (Hint: one will be the angle between the bottom of the square and the bottom of the triangle. Call this one θ.)

Question 3: One of your variables will be your *parameter* that you'll change until you get the maximum area of the triangle. Pick one variable (and try to pick wisely—a bad choice may make the problem harder) and then come up with a formula for the area of the triangle in terms of your variable.

Question 4: With your formula in hand, use techniques you know to find the value of your variable that gives you the maximum area for the equilateral triangle. Be sure to pay attention to the proper range of your parameter.

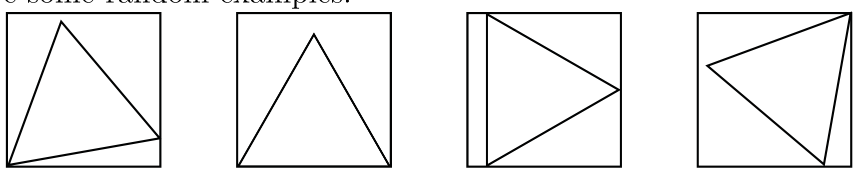
Question 5: So, what is your answer? What triangle gives the biggest area? Find a folding method that produces this triangle.

Follow-up: Your answer to Question 5 can also give a way to fold the largest *regular hexagon* inside a square piece of paper. Can you see how this would work?

Handout
What's the Biggest Equilateral Triangle in a Square?

In this activity your task is to find the biggest equilateral triangle that can fit inside a square of side length 1. (Note: an equilateral triangle is the triangle with all sides of equal length and all three angles measuring 60°.) The step-by-step procedure will help you find a mathematical model for this problem, and then to solve the optimization problem of finding the triangles position and maximum area.

Here are some random examples:



Question 1: If such a triangle is maximal, then can we assume that one of its corners will coincide with a corner of the square? (Hint: The answer is yes. Explain why.)

Question 2: Assuming Step 1 above, draw a picture of what your triangle-in-the-square might look like, where the common corner of the two figures is in the lower left. (Hint: see one of the four examples above.) Now you'll need to create your model by labeling your picture with some variables. (Hint: Let θ be the angle between the bottom of the square and the bottom of the triangle. Let x be the side length of the triangle.)

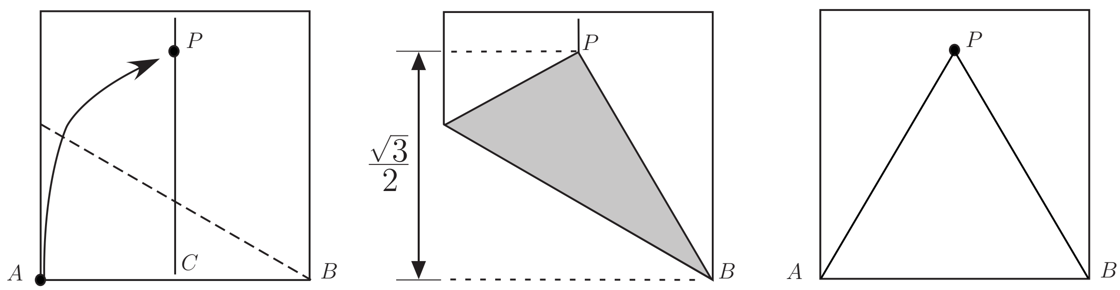
Question 3: Come up with the formula for the area of the triangle in terms of one variable, x. Then, find an equation that relates your two variables, x and θ. Combine the two to get the formula for the area of the triangle in terms of only one variable, θ. (Hint: your last formula will be $A = \frac{\sqrt{3}}{4}\sec^2\theta$.)

Question 4: What is the range of your variable θ? Explain. (Hint: the range should be $0^\circ \leq \theta \leq 15^\circ$.)

Question 5: Most important part: With your formula and the range for θ in hand, use techniques of optimization to find the value of θ that gives you the maximum area for the equilateral triangle. Also, find the value of this maximum area. (Hint: For simplicity, you may want to express all trigonometric functions in terms of sin and cos).

Solution and pedagogy

Folding an equilateral triangle: There are a number of ways to fold an equilateral triangle in a square. All involve finding a way to produce a 60° angle. Your students might find new and creative ways to do this, but the most common way people discover is shown below. (We assume in these pictures that the side of the original square has length 1.)



The origami "move" here is to take one corner, A, and fold it to the center line (so the paper must have been creased in half first) *while at the same time* making sure the crease you make goes through corner B.[1] We let P be the image of point A under this fold. Then we have that ABP is an equilateral triangle. This can be seen in a number of different ways:

- Let C be the midpoint of AB. Then considering $\triangle BCP$, we have that BP has length 1 (since it is the image of AB) and BC has length 1/2. The Pythagorean Theorem then tells us that CP has length $\sqrt{3}/2$, so $\triangle BCP$ is a 30°-60°-90° triangle. Then creasing AP gives us an equilateral triangle.

- Since BP is the image of AB under the folding, BP has length 1. We can then either say, "Now fold B to the center line in the same way," or "By symmetry," to get that AP has length 1 as well. Thus $\triangle ABP$ is an equilateral triangle.

In the solution pictured here, the length of the side of our triangle is the same as the side of the square. However, if we imagine rotating the triangle counterclockwise a little bit about the point A, we could then expand the sides some and still remain inside the square. So it *is* possible to make a bigger equilateral triangle inside the square.

Pedagogy: Many students will first try to construct a 30°-60°-90° triangle by trying to make the right angle be at a corner of the square. This is not the easiest thing to do, and suggesting that such students try folding the corner inside the square

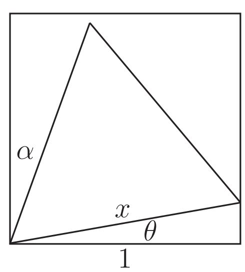
[1] This is a standard origami move: a point p_1 is folded onto a line l, but that is not enough to determine where exactly the crease should be made. So a second point, p_2, is needed, where we make sure the crease line goes through p_2 as well as making p_1 land on l. See the Folding a Parabola activity for more information.

instead can get them over this mental block. Suggesting that they use the 1/2 center line can also be offered.

Oftentimes students overhear ideas from other groups in class, or a good idea gets suggested from one group to another. That's fine, but everyone should write down a proof that their triangle is really 30°-60°-90° or equilateral. Groups should present their proofs to the class so that everyone can see that it can be done in more than one way. Writing up their proofs formally can be assigned individually for homework, if desired. (This should be easy after the group work, but writing things up "for real" is still a very valuable activity.)

Finding the maximal triangle: There are two versions of this handout: one that provides only a frame for the problem, leaving all the details to the students, and one that walks the students through the problem, step-by-step. The solutions are basically the same and presented here in tandem.

For the first question on the handout, the answer is yes. If no corner of the equilateral triangle is on a corner of the square, then the triangle must not be touching one side of the square (since the triangle has three corners and the square has four sides). Assume this is the left side. Then the three corners of the triangle must be touching the three other sides of the square, for otherwise we could make the triangle bigger. Then we can slide the triangle to the left until it touches this left side with one of the corners that touch the top and bottom sides. This puts a corner of the triangle on a corner of the square.

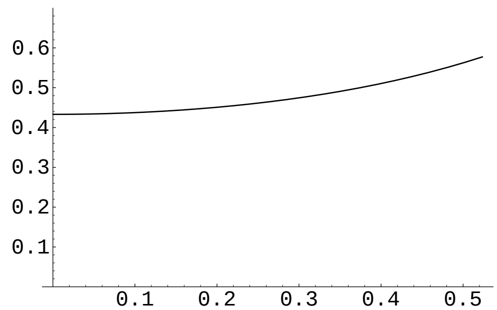


To set up the model, students will need a picture something like the above figure. The base of the triangle (length x) should extend from the bottom left corner to the right side of the square. Then we need to consider the range $0° \le \theta \le 15°$, for if $\theta > 15°$, then we'll have $\alpha \le 15°$ and we'd be in a case symmetric to one with $\theta \ge 15°$. In other words, the symmetry of the square restricts the range of θ that we need to consider.

We need to find a formula for the area A of the equilateral triangle and then try to maximize this formula in terms of θ. (We want to do this in terms of θ, instead of x, because θ is the variable that tells us the position of the triangle in the square.) Since the base of the triangle is x, its height is $(\sqrt{3}/2)x$. So $A = (\sqrt{3}/4)x^2$, but we wanted it in terms of θ. Well, $\cos\theta = 1/x$, so $x = 1/\cos\theta = \sec\theta$. Thus we have

$$A = \frac{\sqrt{3}}{4}\sec^2\theta.$$

We could take the derivative of this and try to maximize it using calculus, but we don't really need to. Since $\cos\theta$ is a decreasing function on the interval $0 \le \theta \le \pi/12$ (we really should be working in radians, after all), we know that $\sec\theta$ is an increasing function on this interval. The same will be true of $\sec^2\theta$, so the maximum value of A will be on the right-most endpoint of the interval, $\theta = \pi/12$. Students can see this by graphing the function $A(\theta)$:



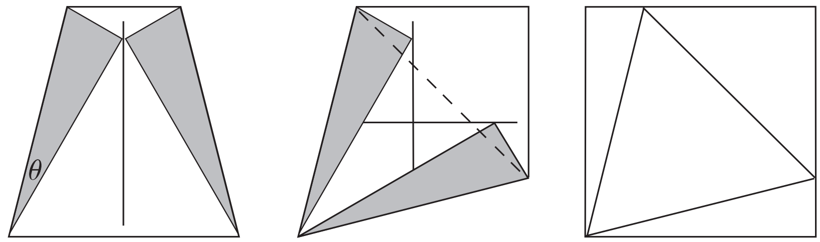
Thus the maximum area is achieved at $\theta = \pi/12 = 15°$. This results in a picture where one corner of the triangle is on a corner of the square and the triangle is symmetric about a diagonal of the square.

Students who do use derivatives to solve this would get

$$\frac{dA}{d\theta} = 2\frac{\sqrt{3}}{4}\sec^2\theta\tan\theta = \frac{\sqrt{3}\sin\theta}{2\cos^3\theta}.$$

Since $0 \le \theta \le 30°$, we know that $dA/d\theta = 0$ only when $\theta = 0$. This means that the area forluma has a critical point at $\theta = 0$. But this is just an endpoint of our interval, so this means that the extreme values of the area A will happen at the endpoints $\theta = 0$ and $\theta = 15°$ (since there are no critical points in between). The question then is, which is a maximum and which is a minimum? We could take the second derivative of A and determine the concavity of the critical point $\theta = 0$, but taking such a derivative looks a little foreboding. Instead we could just check the value of A when $\theta = 0°$ and $\theta = 15°$. Fifteen degrees wins.

Students who do both of these handouts should be able to find a folding sequence for the maximal equilateral triangle. The pictures below serve as such a folding sequence as well as a "proof without words" that it works. (First note that $\theta = 15°$ in the left-most figure.) This folding sequence proof was developed by Emily Gingras, Merrimack College class of 2002.

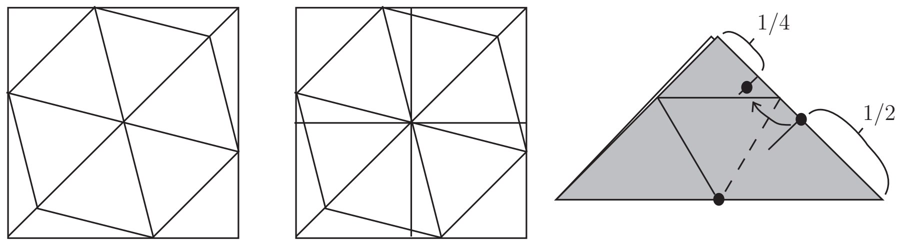


Pedagogy: Students familiar with the classic "fenced-in pen along a side of a barn" or "box folded out of a sheet of cardboard" calculus problems should see right away that our maximum equilateral triangle problem should be solvable using similar methods. However, the model for our problem is very different from those classic ones, and most students find it very challenging to set up the model properly. The hard and subtle part is making sure that you can parameterize the problem with a variable that tells you the triangle's position in the square. The best way to do this seems to be with an angle, and thus a formula for the triangle's area must be found in terms of this angle. In any case, this problem is at the right level of what calculus students learning optimization problems *should* be able to solve. But the value in this activity is for the students to sharpen their mathematical modeling skills, so the instructor should resist giving any more hints than those already given in the handout. Also, students should be encouraged to explore whatever avenue they choose to give a correct proof, be it a numerical, graphical, or analytical approach.

However, not all instructors will want to leave the details of such an activity entirely open. The second version of the optimization handout is for those who would like their students to see the proper procedure for such a problem and work out the details themselves. The format and pacing of this handout follows a suggestion by beta-tester Katarzyna Potocka of Ramapo College of New Jersey.

It can also be valuable to do this activity in a geometry course to emphasize the interconnections between mathematical disciplines. Typically, math major undergraduates in an upper-level geometry course will claim to have forgotten all of calculus, making this all the more worthy to do.

Follow-up activity: If you think about how a maximal regular hexagon would be inscribed in a square, as in the pictures below, and make horizontal and vertical half-way creases, you can see that one quarter of the square is exactly like the crease pattern for the maximal equilateral triangle. Therefore the folding method for the triangle can be modified to give a maximal hexagon. The far right figure below abbreviates such a method.



Of course, these questions can be asked for folding any regular polygon inside a square, and while proving maximality gets more complicated, it's not beyond an undergraduate's means and can make good extended projects. Below are figures that show a way of proving the maximal hexagon case. Let θ be the angle it makes with the bottom edge of the square (whose side length is, again, 1) and let x be the length of a side of the hexagon. The hexagon is made up of six equilateral triangles, which makes it easy to compute the area of the hexagon: $A = 6\times$(area of one triangle)$= 6(x/2)(\sqrt{3}/2)x = (3\sqrt{3}/2)x^2$. But we want to maximize this with respect to θ.

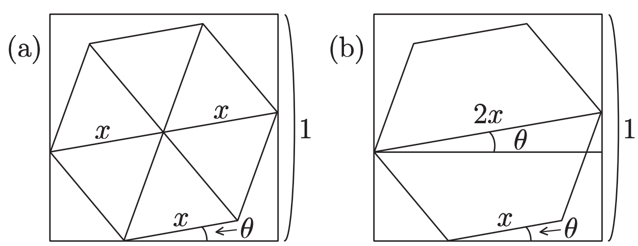


Figure (b) shows how we can do this. The diameter of the hexagon is $2x$, and if we assume that two opposite corners of the hexagon will be touching the left and right sides of the square, then we can form a right triangle from one of these corners (the left one, in the figure) with base of length 1 and hypothenuse one of the diagonals (length $2x$). Since the bottom of this triangle is parallel to the bottom of the square, and the hypothenuse is parallel to the bottom of the hexagon, we know that the base angle in this right triangle will be θ. Thus $\cos\theta = 1/2x$, or $x = (1/2)\sec\theta$. Thus the area of the hexagon is $A = (3\sqrt{3}/8)\sec^2\theta$.

To maximize this, we need to find the range of θ we need to consider. The symmetry of the hexagon shows us that $0° \leq \theta \leq 15°$ is all we need to consider. Like the triangle case, the largest endpoint of this interval, $\theta = 15°$, gives the largest area. This will make one of the diagonals of the hexagon lie along a diagonal of the square.