PBS Mathline®

Algebraic Thinking Math Project

Looking Through the Algebraic Lens
Grades 6-8

Overview

Algebraic Thinking Focus

Teachers in grades 3-8 should make it a goal to foster the development of algebraic thinking in every strand of their mathematics curriculum. Topics in each strand offer opportunities to develop a basic understanding needed for success in the formal study of algebra.

Overview of the Lesson

This video includes two activities that focus on the development of algebraic thinking with topic traditionally considered outside the patterns, function, and algebra strand of the mathematics curriculum. The first lesson focuses on topics from the number and operations strand as well as the measurement strand. The second lesson is a portion of a problem-solving activity focusing on algebraic thinking applied to measurement and geometry topics.

Part I: How Much is a Pint?

Part II: The Skeleton Tower Problem

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Part I: How Much is a Pint?

Lesson Objective

Students will estimate capacity and the relationship between several unusally shaped containers. They will verify relationships, express the relationship using variables, and apply the expressions to determine equivalent portions for each standard unit.

Materials

For each group:

- Simple cyclindrical containers with the following capacities: 1 cup, 1 pint, 1 quart, and 1 gallon
- Irregular-shaped containers with the following capacities: 1 cup, 1 pint, 1 quart, and 1 gallon
- Food coloring
- Water
- A pitcher
- Paper towels
- *How much is a Pint?* activity sheet

Procedure

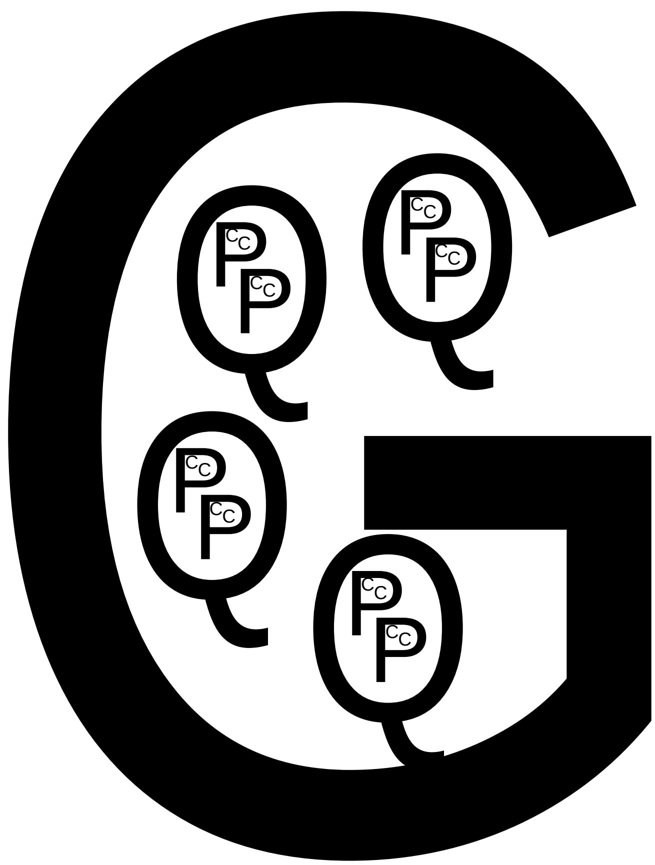
Prior to the Lesson: It would be helpful if students have some experience using variables to write equations for known relationships. For example, they might be able to express the relationship between an inch and a foot as $f = 12i$ and $i = (^1/_{12})f$.

1. **Introduction**: Present the class with the following challenge: For an upcoming class party, the teacher must provide drinks. However, juice and soft drinks come in containers of many different sizes and shapes. The teacher needs help determining how many containers of each size would be needed.

2. **Estimating the Relationships among the Capacities of the Containers**: Show examples of irregular-shaped containers to the students, each labeled with a variable. Tell them to work in their groups to estimate the relationships between the capacities of the containers.

   Example: Write equations to answer the following questions:
   - How many times would the contents of container U (full) fit into container T?
   - What fraction of the capacity of container R is the capacity of container S?

3. **Sharing Estimates and Verifying the Relationships**: Have groups share their estimates. Establish the actual relationships of the capacities by pouring liquid from one container into another. Reveal the customary capacities of the containers (cup, pint, quart, and gallon) and show students the following How much is a Pint visual:



   Have students write equations that show the relationship between ounces, cups, pints, quarts, and gallons. Verify these equations through class discussion.

4. **Using Equations to Solve the Problem**: Have groups use the equations to determine how many gallons, quarts, pints, cups, and ounces will be needed for a party if all of the students in the class get 1 pint of liquid. Have groups share and verify their results.

Examples student work are at the end of this lesson.

Mathematically Speaking

"Many teachers find it helpful to introduce algebraic expressions as models of quantities in contextual situations...By working on problems like these, students can gain experience with symbolic algebraic representations of situations. Moreover, they can see that different symbolic expressions can be produced to represent the same situation."

NCTM Principles and Standards for School Mathematics (Draft), 1998

Students need numerous opportunities to make estimates related to measurement in order to fully grasp this concept. Many middle school students cannot estimate lengths or distances very well and they are generally weaker with estimates of capacity. Thus, the first part of this activity was designed to provide a novel yet realistic context through which students could share and refine strategies related to such estimates.

Symbolic representation was introduced by asking the students to express the estimated relationships between the capacities of the various containers in the form of equations. Symbolic representation is an intregral part of algebraic thinking. As shown in this activity, symbolic representation can be used in making predictions in a real world situation.

Linking variables to models makes the concepts related to variables easier to grasp and retain. For example, with the containers in front of them, it was easy for students to see that if $4S = R$, then $S = \frac{1}{4} R$. The concept of reversibility is easily applied to work with variables when students can see the concrete objects they represent. Similarly, since the containers were available to support their thinking, they had no trouble determining that if $2T = S$ and $4S = R$, then $8T = R$. This realization lays the foundation for using substitution and the transitive property with "bare" symbols. For students who disagree or are struggling, the models may demonstrate these and other conjectures and / or conclusions. All these factors contribute to making students more comfortable with variables and equations in the future and eventually when no models are available.

Once the capacities and relationships between the containers are known, repeating the activity reinforces lessons from the first part of the activity, emphasizes relationships among customary units of capacity, and provides another context for using variables and equations with real world meaning.

Finally, the last part of the activity serves as an informal assessment of student understanding. The students in the video were able to write equations using variables, use them to express 22 pints as 11 quarts, 44 cups, 352 ounces and 2 ¾ gallons, and explain their thinking. The variety of correct methods used to arrive at those equivalents described by the students in the video provided evidence for the teacher that they grasped the relationship between basic units of capacity. Their work also demonstrated an ability to understand and use variables and equations related to those units. Student explanations provide alternate strategies for other students who might need them.

Extensions

- Each group can gather four unusually shaped containers, measure the capacity of each, and write equations to show the relationship of their capacities. Then groups can exchange sets of containers and repeat the steps carried out in this lesson. Each group should take the new set of containers and do each of the following steps:

    1. Estimate the relationships between the capacities of the containers, using equations containing variables.
    2. Pour liquid and use standard units of capacity to establish actual relationships between the capacities of the containers.
    3. Make any needed adjustments in the equations.
    4. Determine how many times each container would have to be filled to equal a given amount of liquid.

    The original group of students should check the work of their "students." These equations may involve fractions or decimals unless the containers are carefully chosen.

- "Fill 'er Up" is a lesson from PBS MATHLINE's Middle School Math Project (MSMP). "Fill 'er Up" would complement this lesson nicely. It involves students interpreting, predicting, and sketching graphs of functions related to filling containers of different shapes.

Technology Connections

Using Excel or a similar software application, help students create a spreadsheet that displays equivalent amounts of liquid using the basic units of capacity.

- A business uses a fifty-gallon drum container to store liquid. Create a spreadsheet to display equivalent amounts of the liquid measured in quarts, pints, cups, and ounces for 1 to 50 gallons.

Equivalents based on the Number of Gallons

# Gallons	#Quarts	#Pints	#Cups	#Ounces
1	A2*4	A2*8	A2*16	A2*128
2				
3				

- Grocery stores often have milk delivered in square crates containing 25 quarts. Create a spreadsheet to display equivalent amounts of the liquid measured in pints, cups, ounces, and gallons for 1 to 25 quarts.

Equivalents based on the Number of Quarts

#Quarts	#Pints	#Cups	#Ounces	#Gallons
1	A2*2	A2*4	A2*32	A2/4
2				
3				

If your students have experience with spreadsheets you can work through the first spreadsheet as a class and then ask students to create the second spreadsheet on their own, basing the formulas on the data in the first column (# of quarts). Other students could base their formulas on the number of pints, cups, or ounces as well.

Resources

Articles:

❖Brinker, Laura. "Using Recipes and Ratio Tables." Teaching Children Mathematics. December 1998, 218-224.

❖Curcio, Fran, et.al. "Exploring Patterns in NonRoutine Problems." Mathematics Teaching in the Middle School. February, 1997, 262-269.

Other:

❖Billstein, Rick, and Jim Williamson. MathThematics Program. Evanston, IL: McDougal Littell, 1999.

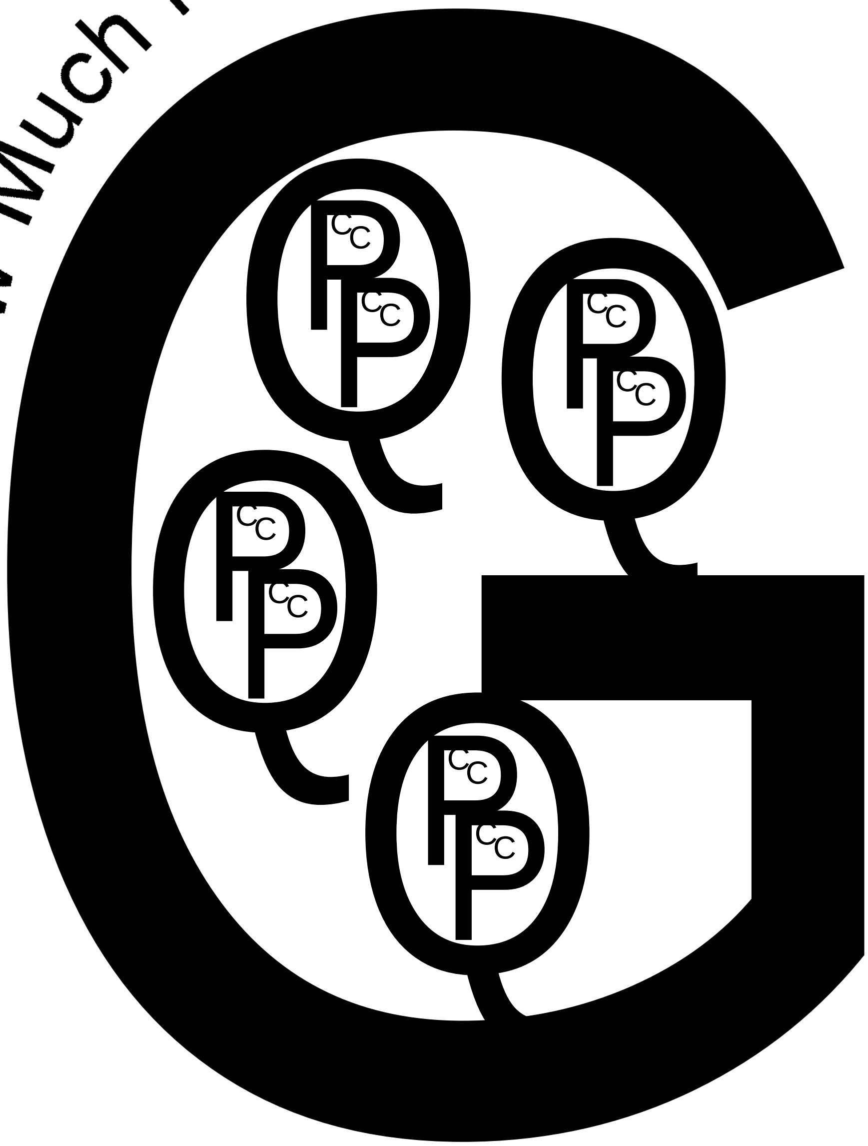
❖Unifix Cubes
http://www.edumart.com/didax/
http://www.familydiscount.com/mathmanipulativesa.htm
http://st4.yahoo.net/eastside/98309.html

❖*Principles and Standards of School Mathematics:* (Draft),National Council of Teachers of Mathematics. Reston, VA: NCTM, 1998.
http://www.nctm.org

How Much is a Pint?

Name:______________________

Date:________



1 cup = 8 fluid ounces

How many fluid ounces are there in 1 pint? 1 quart? 1 half gallon? Show all of your work.

1pt =_________fl oz   1 qt = _________fl oz

1/2 gal =_________fl oz   1 gal = _________fl oz

Examples of Student Work

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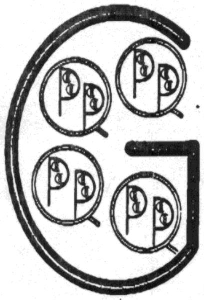
Meredith K.



1 G = 4 Q

1 G = 8 P 1 gal = 128 O

1 G = 16 C

16
× 8
128

1 P = 1/8 G

2 P = 1 Q

4 C = 2 P

4 Q = 1 G

G = gallon
P = pint
Q = quart
C = cup
O = ounce

1 Q = 1/4 G

1 Q = 2 P

1 Q = 4 C

1 Q = 32 O

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Student Work

Capacity Estimates — Justin Heines

1/26/99

container U goes into container S about 5 times
container U goes into container T 2 times

S • 3 = R
T • 7½ = R
U • 15 = R
1/5 S = U
1/2 T = U
1/15 R = U
1/3 R = S
2/5 S = T

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Part II: The Skeleton Tower Problem

Lesson Objective

Students will model a situation, look for patterns, find a general rule, and express the rule using a variable.

Materials

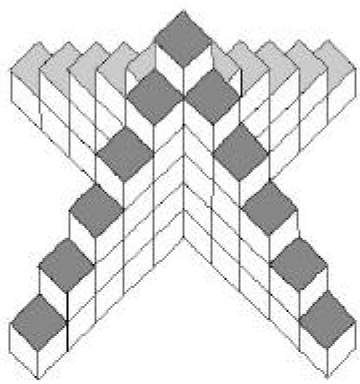
For each group:

- *Skeleton Tower* activity sheet
- Manipulatives such as multi-link or snap cubes
- Chart Paper
- Markers

Procedure

Prior to the Lesson: It would be helpful if students have some experience finding patterns and expressing rules with variables.

1. **Introduction:** Present the class with the *Skeleton Tower* problem as shown below:



1. How many cubes are needed to build this tower?
2. How many cubes are needed to build a tower like this but 12 cubes high?
3. Explain how you worked out your answer to question #2.
4. How many cubes are needed for a tower of any height? Make a rule using a variable and explain your thinking.

2. **Solving the Skeleton Tower Problem**: Have students work in groups to solve the Skeleton Tower problem. Provide each group with copies of the activity sheet, chart paper, markers, and manipulatives. Allow each group to model the problem using any available materials. Students should include any patterns that they notice on the charts they create. Also, students should state a general rule (if they determine one).

   When all groups have completed the assignment, display the charts and have each group explain their strategies and results. Address any incorrect solutions and compare alternate forms of the rule to ensure students realize they are equivalent.

Mathematically Speaking

"Through our experiences of designing instructional activities that integrate non-routine, nontraditional problems, we found a meaningful way to involve students in exploring and formalizing patterns, conjecturing about patterns they identify, verbalizing relationships between and among elements in the patterns, and eventually generalizing and symbolizing the relationships-all essential components of algebraic thinking."

Fran Curcio, et al. in *Mathematics in the Middle School*, February 1997

While experts do not always agree on all the components of algebraic thinking, the importance of exploring patterns in the development of algebraic thinking is unchallenged. Middle school students need numerous experiences with patterns in rich problem-solving contexts to foster algebraic thinking.

When choosing tasks to use in instructional activities, select those that can be solved or viewed in more than one way. The Skeleton Tower problem often elicits several different methods for discovering the general pattern. By listening to other students' strategies and conclusions almost every child in the class has the opportunity to add to his/her "mathematical toolbox." With the Skeleton Tower problem, one group in the video classroom counted the cubes in one section (15 for a 6-high tower), multiplied that by 4 since there were four sections (sub-total = 60 for a 6-high tower), and then added the cubes in the middle stack (total = 66 for a 6-high tower). They repeated this procedure for a 12-high tower (66 cubes per section, 264 cubes as sub-total, and 276 total cubes) and then compared their results. Working backwards, they found numerical patterns and determined that the rule for the nth tower was $[n + (n - 1)]n$. Another group put the walls together in pairs to form rectangles (two 6 x 5 rectangles for the 6-high tower). Then, they multiplied by two since there would be two such rectangles (sub-total = 60 for a 6-high tower) and added the middle stack (total = 66 for the 6-high tower). Their rule was expressed as $2[n(n - 1)] + n$. Other versions of the rule are also possible.

It is crucial that students understand that all the correct expressions are equivalent. Most students will be convinced that this is true by comparing the table of values generated for each such expression (either by hand and/or using a calculator). When the function values are the same for every value of n, many students become confident that the rules are equivalent. However, there are numerous middle school students who need to evaluate each expression for one or more values of n before they are convinced. In such cases, another student can evaluate each suggested form of the function rule at the board or overhead by substituting a particular value (10, e.g.) for each n in the rule his/her group derived. Students can compare the function values and repeat the process for other values of n, if needed. For other students, seeing that the graphs for all the equivalent expressions are identical is the most convincing.

There are some groups of students in middle school for whom using symbolic manipulation to simplify the various forms of the rule is appropriate. For example, using the distributive property, a student may rewrite $[n + (n - 1)]n$ as $n^2 +(n^2 - n)$ and then as $2n^2 - n$. Similarly, $2[n(n - 1)] + n$ may be rewritten as $2(n^2 - n) + n$ and then as $2n^2 - 2n + n$ and finally as $2n^2 - n$. However, this symbolic manipulation should not be used until you are confident that your students have reached the appropriate level of abstract thinking to make this process meaningful.

Resources

Other MATHLINE lessons that provide Algebraic Thinking Across the Mathematics Curriculum in Grades 3-8

The following lessons are found in the Elementary School Math Project (ESMP) or the Middle School Math Project (MSMP). The videos and support materials can be obtained through PBS. For some lessons you will need to make adaptations to emphasize the development of algebraic thinking.

Strand	Project	Lesson Title	Description	Connection
Measurement	ESMP	Bubble Mania	Measuring the diameter, circumference, and area of circles made by a bubble print	Deriving algebraic expressions and formulas
Number and Operations	ESMP	It Takes Ten	Estimating and measuring through a variety of lab experiences	Proportional Reasoning
Number and Operations	ESMP	Soak It Up	Comparing products to determine the best value	Proportional Reasoning, Unit Rates
Number and Operations	ESMP	An Apple a Day	Making estimates to analyze the number of apples per acre and visualizing the magnitude of one million	Proportional Reasoning, Unit Rates
Number and Operations	ESMP	Food for Thought	Identifying and comparing unit costs of given items	Proportional Reasoning, Unit Rates
Patterns, Functions, and Algebra	ESMP	Peddling Petals	Creating, extending, and describing arithmetic patterns found in paper flower designs	Patterns
Patterns, Functions, and Algebra	ESMP	Struts n' Stuff	Identifying the relationship between the number of sides in a regular polygon and the number of struts needed to make each polygon rigid	Patterns and Variables
Patterns, Functions, and Algebra	MSMP	Aw Chute	Determining and comparing the rate of descent of various student constructed parachutes	Proportional Reasoning, Rate
Patterns, Functions, and Algebra	MSMP	In a Heartbeat	Using scatterplots to determine the correlation between heartbeats per minute before and after aerobic exercise	Proportional Reasoning, Rate
Geometry	MSMP	Let's Face It	Identifying, describing, and constructing the five regular polyhedral	Patterns and Variables
Measurement	MSMP	How Many Noses Are In Your Arm	Applying proportional reasoning to determine the length of the Statue of Liberty's torch-bearing arm	Proportional Reasoning
Patterns, Functions, and Algebra	MSMP	Fill 'er Up	Interpreting, predicting, and sketching graphs of related functions as applied to the shapes of bottles	Functions
Patterns, Functions, and Algebra	MSMP	The Great Race	Constructing number patterns generated from a tortoise and the hare race and using them to generate a graph of the situation	Patterns and Functions
Statistics	MSMP	Something Fishy	Applying proportional reasoning to the capture-recapture statistical procedure	Proportional Reasoning

Ideas for Online Discussion

Ideas are linked to the Principles and Standards for School Mathematics.

1. Give a specific example of a lesson traditionally categorized in the number and operation strand or the measurement strand that you have used in your classroom with an emphasis on the development of algebraic thinking.

2. Give a specific example of a lesson traditionally categorized in the geometry strand that you have used in your classroom with an emphasis on the development of algebraic thinking.

3. What are some problem-solving activities at your grade level(s) that promote the development of algebraic thinking across the strands of your mathematics curriculum?

4. Which strands of the mathematics curriculum do you find it especially challenging to incorporate an emphasis on the development of algebraic thinking? Why?

Skeleton Tower