PBS Mathline®

Algebraic Thinking Math Project

Up, Up, and Away

Algebraic Thinking Focus

Understanding slope as a rate of change is an important part of the study of functions, which in turn provides the basis for much of the study of algebra. Students need concrete experiences in diverse and meaningful contexts to develop an understanding of the concepts of slope and function.

Lesson Objective

Students will create graphs from a spreadsheet, discover linear relationships, and explain the real world meaning of slope.

Overview of the Lesson

Students measure circumference and flight time for deflating balloons. They create a spreadsheet of the data and create graphs using pairs of actual and projected measures from the experiment. They then examine the graphs to identify linear and non-linear relationships. They analyze the slope of the linear relationships and interpret the meaning of their graphs.

Materials

For the class:

- Balloons (12" diameter) with small stems
- Calipers or alternate set-up for measuring diameter of inflated balloons
- Tape measures
- Stopwatches
- Computer / pencil and paper for recording data

Procedure

Prior to the Lesson: Students should have some experience with graphing and interpreting graphs for sets of data. They should have some familiarity with functions and the concept of slope. It would be helpful if they had some prior experience with spreadsheets and graphing on computers.

1. **Introduction**: Blow up a balloon in front of the class. Release the balloon and let it go until it completely deflates. Then pose the following challenge to the students:

   Design an experiment in which you use mathematics to discover, describe, and predict what happens when you allow balloons of different circumferences to completely deflate.

   To foster the discovery component of the lesson, you should not discuss with students the results they might expect or conclusions they might draw. However, you should emphasize that they may be unable to find relationships that actually exist if their procedures are inconsistent or their measurements inaccurate. In designing the experiment, students should suggest factors which may affect their data gathering and ways to maximize the accuracy of their measurements.

2. **Gathering Data**: Have students work in small groups. Each group should inflate their balloon to a circumference of 40, 50, 60, 70, 80, 90, and 100 centimeters for successive trials.

   A student records the data for each trial and each group keeps a record of their data as well. If it is possible, a student may be designated to enter each trial into a computer spreadsheet for easier averaging of the data.

3. **Creating the Spreadsheet**: Through class discussion, establish column headings for the spreadsheet. The following categories were used in the video classroom:

Measured Circumference in Centimeters	Measured Diameter in Centimeters	Radius Projected from Diameter in Centimeters	Projected Surface Area in Square Centimeters	Projected Volume in Cubic Centimeters	Time to Deflate in Seconds

   In the first column, students enter the measures 40-100, incrementing by ten for each new row. In the second column students enter the means of their measured diameters. Appropriate formulas should be used to generate the projected radius, surface area, and volume for each measured circumference in the next three columns. The mean times needed for the balloons of each size to deflate are entered in the final column.

In the video class, it was necessary to copy three columns from the spreadsheet *(projected radius, measured diameter, and measured circumference)* onto a separate page in a different order to allow the creation of particular scattergrams. This may not be necessary, according to the spreadsheet software your students use.

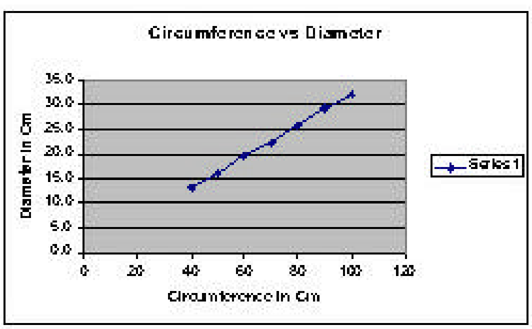
4. **Graphing and Analyzing the Graphs**: In the video class, each group of three students was randomly assigned to explore two of the relationships listed below.

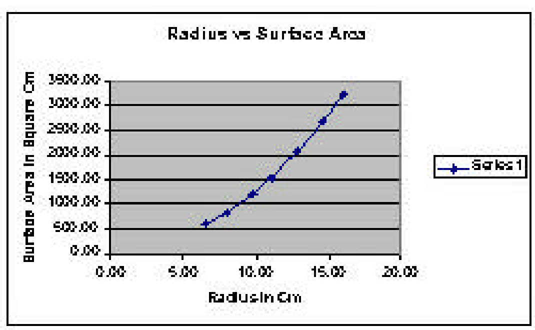
Independent (X) Variable	Dependent (Y) Variable
Radius	Diameter
Diameter	Radius
Circumference	Time
Diameter	Radius
Radius	Surface Area
Circumference	Diameter
Diameter	Circumference
Radius	Volume
Radius	Time

Using the computer, each group of students creates the graphs (scattergrams) of two pairs of variables with the following objectives:

1. Determine whether there is a linear relationship between the variables. (The assignments were made so that only one of the relationships would be linear but the students were not told this information.)
2. If there is a linear relationship, determine the slope. Be prepared to explain how the slope fits with what you know about the experiment and about geometric relationships.

Example:





5. **Interpreting the Linear Relationships:** Each group of students displays the scattergram of a linear relationship displaying data gathered in the experiment and the students explain the slope they determined and its meaning.

In the example shown above, the students determined that the relationship of circumference to diameter is linear and the slope of the graph of that relationship is about 1/3. They explained that as the balloon became larger in the experiment, the diameter of the balloon grew about one-third as fast as its circumference. This fit with what they had learned about spheres: circumference = π (about 3) x diameter, so diameter = circumference/π or diameter = about 1/3 x circumference. They also determined that the relationship of radius to surface area is not linear and explained how that fit with their observations in the experiment and their knowledge about spheres.

6. **Conclusion**: The teacher facilitates a discussion about the general findings of the experiment, predictions students might make from their data, and the value of such an experiment.

Mathematically Speaking

"It is particularly important for middle grades students to gain a deep understanding of rate of change and how it relates to lines and slopes of lines. Students may have difficulty, [but] ...teachers can help overcome this problem by giving students experiences with problem situations in which change is occurring and by helping students reflect on the connections among the actual change, its visual representation on a graph, and the constant m in linear equations y = mx + b."

Standards and Principles for School Mathematics (Draft) 1998

The balloon experiment offers a motivating context for students to gather and analyze data, explore relationships between variables, investigate slope in linear graphs, connect algebra with the real world, and connect algebra and geometry in meaningful ways.

When students set out to gather data in order to identify relationships between the variables, it is important that the data is as accurate as possible. If their procedure is inconsistent or their measurements inaccurate, students may be unable to find relationships that actually exist or the nature of those relationships may be unclear. For example, in this experiment the small differences in the amount of time required for a balloon to deflate for successive trials can be problematic. If only one student is timing the trials and that person is a little quick to stop the time on the last few trials, the graph of *diameter* versus *time to deflate* may appear linear. Thus, your students should establish definite procedures for performing the experiment. As shown in the video, it may be helpful to stop after each group has done one trial and re-emphasize or modify procedures.

To get the best data, we found that students should perform the following steps for each trial:

- Each group uses one balloon (with a relatively small neck) for all trials.
- One person measures and one person verifies the balloon's circumference.

- One person measures and one person verifies the balloon's diameter (One way to do this is to place the inflated balloon between two vertical surfaces such as boxes, books, etc., on a table where a measuring tape has been taped to the table top.) If calipers are available, they can be used to measure the diameters.
- The student holding the balloon aims it towards open space and counts down, "3-2-1-Go," and then says, "Stop," when he/she judges the balloon to be completely deflated.
- Several students use stopwatches to time each trial, and the times are averaged. For any trial a timer may remove his/her time from the pool if he/she feels it was flawed.

It should be noted that there are advantages to having each group of students determine their own procedures and create a spreadsheet based solely on their own data. Ideally, the experiment could then be repeated with any changes in procedure determined by the students. This lesson guide is meant to provide tips for producing a data set that closely reflects reality so that the emphasis of the lesson can be on the intended objectives related to functions and slope. See Coes (1994) for a discussion of related issues.

When entering the data in the Class Data Sheet, use the mean of the values obtained by all the groups in the columns *measured diameter* and *time to deflat*e. This technique will increase the likelihood of graphs that display or strongly suggest the actual relationships between the given variables. When all groups use the same set of data, it also makes it easier for students to focus on the shape and meaning of the graphs instead of the values themselves.

The Class Data Sheet from the video classroom is at the end of this guide.

Creating scattergrams using Excel or another spreadsheet application allows students to change the scale on one or both axis(es) with ease and allows students to concentrate on the relationships between variables instead of the mechanics of graphing. If computers or such software are unavailable, the students can construct the scattergrams on transparencies and share with the class using an overhead projector.

Students need to be aware that many software programs that are used to produce scattergrams from spreadsheets automatically use the left-hand column of selected data as the independent variable and the right-hand column as the dependent variable; furthermore, often this feature cannot be overridden. Thus, if the column headings suggested above are used to create the Class Data Sheet, it might be impossible to create a scattergram displaying *measured diameter* as the independent variable versus *measured circumference* as the dependent variable. This situation existed in the video classroom and is the reason three reordered columns of data were added to the spreadsheet.

When students interpret graphs based on experiments they perform, concepts related to functions are easier to grasp and retain. In this lesson students who understand that a linear relationship exists when the rate of change in one variable is directly related to the rate of change in another variable can apply that knowledge to their observations in the balloon experiment. For example, from the scattergram, students easily identified the relationship between *radius*(x) and *diameter*(y) as a linear function and determined the value of the slope as 2. This made sense to students: since they knew that the diameter of any circle or sphere is always twice its radius, it followed that for any change in the radius of the balloon, the change in the diameter of the balloon is twice as much.

Similarly, it was easy for students to understand that the circumference of the fattest part of the balloon did not grow at nearly as quickly as the volume (amount of air inside the balloon). Thus, they were not surprised that the graph of *circumference* (x) versus *volume* (y) did not display a linear relationship.

These interpretations of the linear graphs in terms of both geometric relationships and also the balloon experiment are the heart of this lesson. It is these connections between algebra and geometry and the use of algebra to interpret and predict real world phenomena that should be emphasized.

Having students share their findings is another important part of the lesson. Since each group examined only two pairs of variables, listening to their classmates greatly expands the learning opportunities of the lesson. Also, it is very valuable for students to contrast the graphs of the same two variables when the independent and dependent variable are reversed. For example, students should notice that the slope of *diameter* (independent) versus *circumference* (dependent) is π whereas the slope of *circumference* (independent) versus *diameter* (dependent) is $1/\pi$. Realizing that the slopes of two such graphs are reciprocals is an important concept in the formal study of algebra.

Related Research Findings

Students develop a better understanding of a concept when both examples and non-examples of that concept are used (Geddes and Fortunato, 1993; Fuys and Liebov, 1993). The best way to help students understand linear functions is to see them in the context of both linear and non-linear functions. In this lesson, each group of students has been assigned both a linear function and a non-linear function to graph and analyze.

Students are less successful at interpreting graphs than they are at interpreting tables, and less successful at interpreting graphs and tables than they are at reading them (Carpenter, 1981). Identifying a graph as a line and determining the slope of the line require students to go beyond reading the graph. They must analyze the graph based on readings taken at two or more points on the graph.

Few 13- to 15-year olds could determine the slopes of two parts of the same line (Hart, 1981). We should not assume that students find it obvious that all parts of a line have the same slope is a mistake. Students should talk about the slope of different parts of a straight line and come to the conclusion that all parts have the same slope. When they have the same conversation about the slope of different parts of a nonlinear function, they will be expanding their earlier analysis of linear functions, coming to a different conclusion.

Linking graphs of functions to their functional expressions is an important goal of school mathematics (Kaput, 1989). One activity that promotes linking a graph of a linear function with its equation is Guess My Rule, in which students are presented a set of number pairs and asked to describe the rule that relates the first number in the pair with the second. This is an activity that only about one-third of 13-year olds can complete successfully for elementary linear functions. For example, when asked to find the value of y in the table

x	1	3	4	7	n
y	8		11	14	

when x is 3, only 35% of 13-year olds and only 59% of 17-year olds were successful (Carpenter, 1981). When students see that 8 is 7 more than 1, 11 is 7 more than 4, and 14 is 7 more than 7, they can see that y should be 10 when x is 3, and they may be able to see that y is n + 7 when x is n. Thus, the general relationship can be expressed y = x + 7.

Only 25% of eighth-grade students can recognize the more difficult pattern

A	2	4	6	…	14
B	5	9	13	…	?

(Blume and Heckman, 1997).
To emphasize the concept of slope, students might work with multiplicative patterns such as

x	1	2	3	4	n
y	3		9	12	

that can be represented by the linear function y = 3x. By working with a number of related functions of the form y = mx, and by relating the graphs to the functional representations, students come to understand slope both as the ratio of rise to run in the graph and also as the coefficient of x in y = mx.

Once students can relate tables and their graphs, students are ready to understand the concepts of slope and y-intercept as they relate to tables, graphs, and functional expressions (Davis, 1987). An approach Davis has found successful is described in Kieran and Chalouh (1993).

As students become comfortable thinking of slope as rate of change, they can talk about the slope of a non-linear equation in an informal way. In this lesson, some students graph the radius of the balloon on the horizontal axis and the surface area (determined by a formula) on the vertical axis. Students can see that as the radius changes from 6cm to 10cm the change in surface area is less than when the radius changes from 11cm to 15cm. They can see this visually because the graph is steeper in the right side of the curve. Thus, conversations in which students compare a nonlinear graph for two different parts of the graph, discuss when the change was greatest or least, and tell how they know, are important to interpreting the graph. While students actually determine the slope for particular parts of a linear graph, determining the slope of a nonlinear graph requires concepts of limits and is beyond the scope of this lesson.

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References

Carpenter, T. P., M. K. Corbitt, H. S. Kepner Jr., M. M. Lindquist, and R. E. Reys. Results from the Second Mathematics Assessment of the National Assessment of Educational Progress. Reston: National Council of Teachers of Mathematics, 1981.

Davis, R. B. "Theory and Practice." Journal of Mathematical Behavior 6(1) (1987): 97-126.

Fuys, David J. and Amy K. Liebov. "Geometry and Spatial Sense." Research Ideas For The Classroom: Early Childhood Mathematics. Ed. Robert J. Jensen. New York: Macmillan, 1993.

Geddes, Dorothy and Irene Fortunato. "Geometry: Research and Classroom Activities." Research ideas for the Classroom: Middle Grades Mathematics. Ed. Douglas T. Owens. New York: Macmillan, 1993.

Kieran, Carolyn and Louise Chalouh. "Prealgebra: The Transition from Arithmetic to Algebra." Research Ideas for the Classroom: Middle Grades Mathematics. Ed. Douglas T. Owens. New York: Macmillan, 1993.

Kaput, James J. "Linking Representations in the Symbol Systems of Algebra." Research Issues in the Learning and Teaching of Algebra. Eds. Sigrid Wagner and Carolyn Kieran. Mahwah: Lawrence Erlbaum Associates; Reston: National Council of Teachers of Mathematics, 1989.

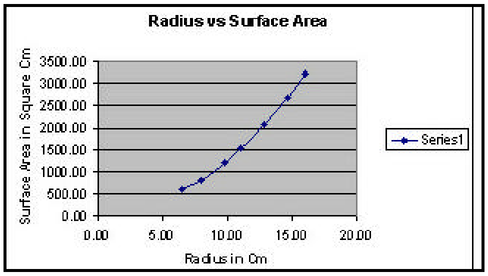
Extensions

- Students can look for linear relationships and explore slope in other sets of data (See McCoy 1997) such as the following:
  - The total number of candies in a bag and the weight of the bag
  - The height of an object and the length of its shadow
  - Measures of height versus arm span for a class of students
  - The number of pennies and the weight of the pennies in grams
  - The amount of water wasted by a leaky faucet over time

  Students should investigate some sets of data for which there is no linear relationship as well.

- If appropriate, students can further examine the non-linear relationships they find between variables in the balloon experiment. They can determine whether those relationships are quadratic or cubic and interpret them in terms of the experiment. They can also make predictions about what would happen with balloons with larger circumferences. Students should have some prior experience with quadratic and cubic functions before they attempt this extension.

  Example:

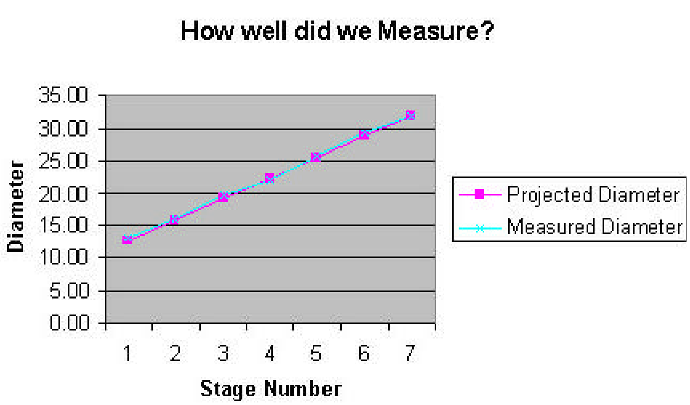


  Students might observe that during the experiment the "skin" of the balloon was increasing at a faster rate than the radius of the balloon since the balloon itself had to stretch in two directions each time the radius increased. This graph also makes sense in terms of the formula for the surface area of a sphere (Surface Area = 4 x π x radius x radius) since each time the radius increases, that increase is squared in the resulting surface area.

Technology Connections

- As a follow-up to the balloon experiment, students can compare the graphs of the measured diameters they obtained with the diameters that would be projected from their measured circumferences using the formula diameter = circumference ÷ π). They can analyze the slopes of the graphs as well.

  Example:



- Students can use the data and graphs from their balloon experiment to predict the time needed to deflate balloons of various circumferences other than those they used in the original experiment. Then students can test their predictions. This activity reinforces the power of algebra as a tool in making real world predictions.

Resources

Articles:

❖Coes III, Loring. "The Functions of a Toy Balloon." The Mathematics Teacher. November 1994, 619-621 and 627-629.

❖McCoy, Leah. "Algebra: Real-Life Investigations." Mathematics Teaching in the Middle School. February 1997, 221-224.

Other:

❖*Principles and Standards for School Mathematics:* (Draft), Reston, Virginia: National Council of Teachers of Mathematics (NCTM)
http://www.nctm.org

Ideas for Online Discussion

Ideas are linked to the Principles and Standards for School Mathematics.

1. In this lesson, students gather data about the time required for balloons of different sizes to deflate. Compared to providing data for the students, this approach obviously requires additional time. When is the extra time worth the benefits for students?

2. When students deal with real world data, the numbers are often "messy." What effect does this factor have on student learning?

3. Computer software was used to plot the scattergrams. Do you think this is an appropriate use of technology? Why or why not?

4. In general, what are the issues related to the use of spreadsheets and graphing software in grades 3-8?

5. The National Council of Teachers of Mathematics recommends that dealing with data is a topic that should receive increased attention. This is the case in many new mathematics programs, including formal courses in algebra. Do you think this increased attention is appropriate? What challenges does this increased emphasis create for teachers in grades 3-8 and how can we address them?

6. How can activities involving gathering, analyzing, and/or graphing data be structured to maximize the development of algebraic thinking?

Class Data

Class Data Sheet for Patterns in Balloon Experiment					
Formulas:			4 x Pi x r x r	4/3 x Pi x r x r x r	
			Projected		
Measured	Measured	Radius	Surface Area in	Projected Volume in	Time to
Circumference in Centimeters	Diameter in Centimeters	Projected from Diameter in Centimeters	Square Centimeters	Cubic Centimeters	Deflate in Seconds
40	13.10	6.55	538.86	1176.50	0.77
50	16.10	8.05	813.92	2184.02	0.86
60	19.60	9.80	1206.26	3940.46	1.29
70	22.10	11.05	1533.61	5648.79	2.19
80	25.70	12.85	2073.94	8883.37	3.38
90	29.20	14.60	2677.29	13029.48	5.46
100	32.00	16.00	3215.36	17148.59	8.83

Radius	Measured	Measured
Projected from Diameter in Centimeters	Diameter in Centimeters	Circumference in Centimeters
6.55	13.10	40
8.05	16.10	50
9.80	19.60	60
11.05	22.10	70
12.85	25.70	80
14.60	29.20	90
16.00	32.00	100