



Building Boxes

(Polynomial and Rational Functions)

Objective

Students will use the graphing calculator to explore general patterns of polynomial and rational functions, and a real-world application of a polynomial function.

Overview of the Lesson

In this lesson, students explore relationships between x-intercepts, factors, and roots of polynomial functions using the graphing calculator. Students also investigate rational functions, identifying the roots and the asymptotes as well as holes in the graphs. Students construct boxes of various dimensions using graph paper, collect height and volume data, and create a scatterplot in order to determine the height of the box with the maximum volume. Students can solve this problem using a graphing calculator or by using their own scatterplots drawn by hand. The use of questioning by the teacher, and the group work of the students are important features of this lesson.

Materials

- graphing calculator overhead unit
- overhead projector

For each group of four:

- graph paper
- scotch tape
- scissors
- markers
- *Polynomial Functions* activity sheet
- *Rational Functions* activity sheet
- *The Box Problem* activity sheet

Procedure

1. **Introduction:** Explain to the students that over the next several days they will be reviewing polynomial functions, exploring rational functions, and solving the box problem. If your students are not used to working in groups, explain to the students the importance of talking with their group members as they explore the concepts.
2. **Polynomial Functions: The connection between the x-intercepts, the factors, and the roots.** Have the students work in groups of four. Each group should have copies of the *Polynomial Functions* activity sheet. Direct students to think about the connections between the three groups of questions. Also remind the students to discuss all their ideas before the group writes their response. Circulate among the groups as the students work on this activity. Encourage students to check their work using their graphing calculators. When most of the groups have completed the activity sheet, have each group exchange their sheets with another group and compare their answers. As they study the response of the other group, they should consider what additions or corrections they would make to their own activity sheet.
3. **Class Discussion of Polynomial Functions:** Facilitate a class discussion to make sure that all students understand the connections between the x-intercepts, the factors, and the roots of polynomial functions. Consider graphing several of these functions so that students can see the connections between the graph and the factors, roots, and x-intercepts. Ask questions to draw the important information from the students.
4. **Polynomial Functions Revisions:** Give the groups a few minutes to revise their activity so that they have all of the important ideas included. Collect the sheets.
5. **Rational Functions:** Introduce this topic by asking students what is a rational function. Lead students to see that a rational function is a ratio with a polynomial in the numerator and in the denominator. Distribute the *Rational Functions* activity sheet and direct the students to explore the five functions. They should note everything of interest about the function including what it looks like, particular points of interest, or important features. Also, if the group has questions about any of the functions, they should be written on the sheet as well. Remind the students to discuss everyone's ideas before deciding what should be written. While the students are discussing these functions, circulate and listen to the groups to determine what points the students understand and what areas they are having difficulty with. Guide students by asking leading questions when appropriate.
6. **Class Discussion of Rational Functions:** Ask questions to obtain all of the important information the students observed about rational functions. They should note that, in general, the zeros of the numerator give the x-intercepts of the graph or the roots of the function, and the zeros of the denominator

give the asymptotes for the function. Make sure that the students understand that the denominator of a rational function can never be equal to zero. Students should also focus on problem # 5, and identify the point of discontinuity. They should realize that while $y = \frac{x^2 - 4x - 5}{x - 5}$ can be reduced to a simpler function, $y = x + 1$, you still must state that $x = 5$ cannot be in the domain of the function. Another point to clarify, if needed, is what happens when there is an asymptote and you graph the function on a graphing calculator. Often students see what they assume is the asymptote itself. Help students to understand that this is not the asymptote, but simply the calculator connecting points which should not be connected. There are at least two ways to keep the calculator from joining those points. One is to change to dot mode under the mode function with the TI-83. The other is to set the window so that the difference between **Xmax** and **Xmin** times some integer value equals 94, the number of pixels along the horizontal direction of the screen. For example, graph $y = \frac{x + 2}{x - 3}$ using a standard window, or

6:ZStandard. Now change **Xmin** to be -8.8, and **Xmax** to be 10. Because the difference, 18.8, times 5 equals 94, the calculator does not connect the points. Setting the **Xmin** at -4.4, and **Xmax** at 5, will also give you a graph where no vertical line appears.

7. **Rational Functions Revisions:** Give the students a brief amount of time to work in their groups to revise their activity sheets. You may wish to collect the sheets.
8. **The Box Problem Introduction:** In the video lesson this problem is used as an application problem for polynomial functions. Present the problem as described on the activity sheet. Explain to the students that they will be working collectively to gather data to determine the dimensions of the box with the maximum volume that can be made from the given sheet of graph paper (39x30 units in the video lesson). Distribute the graph paper, scissors, scotch tape, and markers to each group. Discuss the practical domain for the height of the box given the size of the graph paper, and then assign each group a particular height for their box.
9. **Data Collection:** Have each group construct a box with their assigned height, and then determine the volume of their box. After they have done this, each group should record the height and volume in a table on the blackboard and tape their box up on the blackboard so that other students can see the various shapes.
10. **Data Analysis and Interpretation:** Each group should analyze the data collected by the entire class. Have the students create a scatterplot, and determine the mathematical model that best fits that plot. There are several ways to do this. Some students might use the cubic regression model on the graphing calculator, while others might actually determine the model using the given length and width of the paper, and the formula for the volume of a

box. Once the students have determined the equation of the function that gives the volume of the box for any given height, they can graph this function using the graphing calculator. To find the maximum volume, some students might use the **TRACE** feature of the calculator while others may decide to use the **CALCULATE** option. As a homework assignment, you might have students write up this lab.

Assessment

Assessment in this lesson is ongoing. The nature and direction of the comments made and the questions raised by the teacher stem from observations of what the students are doing and the questions they are asking. Observing the students as they work in groups is an important part of the evaluation process. The group responses made on the activity sheets are also an important aspect of assessment because they allow the students to be part of the evaluation process. The students know where they are with their thinking, and through both the exchange of papers, and through the class discussions, they are able to see and hear the ideas of other groups and then revise their thinking. Asking the students to explain in words the relationship between the roots, factors, and the x-intercepts helps them to solidify their understanding. Through this entire process students take responsibility for their own learning.

The Box Problem can be assigned as a lab to be written up and turned in by each student. This allows for individual accountability for the group work which was done during the class period. Establishing grading criteria ahead of time, and making sure that students understand the criteria is important. For example, for this particular lab the following information might be given to the students when the assignment is made.

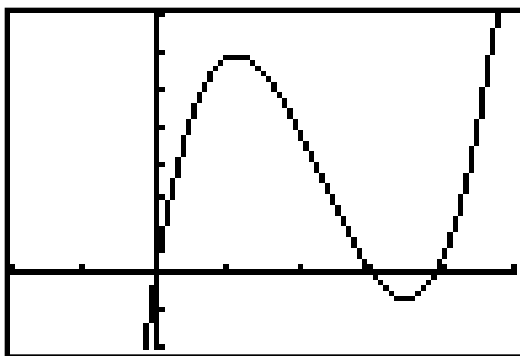
Include in the Lab Report:	Points
The Problem Statement	4
Your Hypothesis	4
Materials Needed	2
Description of the Procedure	5
Data Table	5
Scatterplot	10
Analysis of Data	10
Conclusion	10
TOTAL Points Possible	50

Extensions & Adaptations

- Students can explore multiple roots, both even and odd. They can also investigate end behavior for both even and odd polynomial functions.
- Students can study horizontal and oblique asymptotes. This would be an interesting extra credit project for students looking for a challenge.
- What are the dimensions of a cylindrical can of optimal design? Optimal design means that the can has a minimum surface area for any given volume. (Hint: Start by exploring the dimensions for a can with a specific volume.) *Answer: Any can where the diameter of the can is equal to the height.*

Mathematically Speaking

This lesson offers a wealth of opportunities to connect patterns found in tables, graphs, and equations of polynomial and rational functions. The concepts of range and domain are important to the discussions for a number of reasons, particularly with regard to rational functions where students had to know that x could not be equal to any value that would give the denominator of the function a value of zero. In the box problem, students might discuss the idea of a practical domain for a function and a theoretical domain. For example, $f(x) = (39 - 2x)(30 - 2x)x$ is the function which gives the volume of the box, $f(x)$, in terms of the height of the box, x .



X	Y1	X	Y1
0	0	7	2800
1	1036	8	2576
2	1820	9	2268
3	2376	10	1900
4	2728	11	1496
5	2900	12	1080
6	2916	13	676
$Y1 = (39 - 2X)(39 - 2X)X$			
X	Y1	X	Y1
14	308	19	-152
15	0	19.5	0
16	-224	20	200
17	-340	20.5	451
18	-324	21	756
19	-152	21.5	1118
20	200	22	1540
$Y1 = (39 - 2X)(39 - 2X)X$			

When students plot the volumes to create their scatterplot, they are using values of x such that $0 \leq x \leq 15$, because this is the practical domain for this situation. However, if the students plot and examine the function $f(x) = (39 - 2x)(30 - 2x)x$, they can see that the function certainly continues for values greater than 15 and for values less than 0. The theoretical domain for this function is the set of all real numbers. Students could be asked to think about the physical meaning of the x -intercepts. Whenever students use mathematical models to represent real-world situations, they need to consider the meaning of the practical and theoretical domain.

Tips From Ellen

Assessment

Students need to see a target in order to hit it. A useful strategy in any classroom is to provide and use general rubrics for content area goals, such as the following example for problem solving

General Rubric for Problem Solving	
Exemplary	<ul style="list-style-type: none"> • Contains a complete and accurate response that addresses all of the important elements of the question • Uses an effective strategy and a clear line of reasoning to solve the problem, including examples and counterexamples as appropriate • Includes clear and well labeled tables, diagrams, and models as appropriate to support the solution • Uses clear and precise language and mathematical notation appropriate to the task and intended audience; explanations are coherent and elegant • Goes beyond the requirements of the problem
Acceptable	<ul style="list-style-type: none"> • Contains a complete and accurate response • Identifies and carries out a clear strategy; the line of reasoning can be followed • Visuals are provided as appropriate • Language and mathematical notation are accurate
Approaching	<ul style="list-style-type: none"> • Contains an accurate response which may fail to address some element of the question • Contains some supporting arguments, but the line of reasoning may omit steps or be muddled • Visuals are lacking or inappropriate or unclear • Minor inaccuracies in language and mathematical notation may be present but do not interfere significantly with understanding
Attempted	<ul style="list-style-type: none"> • Fails to address important elements of the question • Uses inappropriate strategies or inaccurate reasoning or line of reasoning cannot be followed • Visuals are lacking or incorrect • Major inaccuracies in language and mathematical notation are present

A rubric of this type can be shared with students and posted in the room. It can then be used in whole or part for self assessment, peer assessment, or teacher assessment in both formative and summative ways. For example:

You're going to be writing frequently in mathematics class this year. Using the rubric we've just discussed, evaluate this example of a student's work from last year on a similar assignment. What level would you assign to this work? What would make it better? Keep these thoughts in mind as you write your paragraph explaining how to solve a linear equation.

When you exchange papers, I'd like you to particularly pay attention to the line in our rubric that addresses the use of language and mathematical notation and provide feedback to your partner on that point.

Some teachers have found it effective to post exemplary student work in the classroom as additional reference points. Others literally make an enlarged target, with the "exemplary" level of the rubric as the bulls eye, "acceptable" the next ring, and so on, and post an example of student work at each level.

Resources

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 9 - 12 : Algebra in a Technological World*. Reston, Virginia: National Council of Teachers of Mathematics, 1995.

National Council of Teachers of Mathematics. *Mathematics Assessment, Myths, Models, Good Questions and Practical Suggestions*. Reston, Virginia: National Council of Teachers of Mathematics, 1991.

Internet Location: <http://tqd.advanced.org/2647/algebra/functype.htm>

This site allows students to explore more advanced types of functions that make a nice extension for this lesson.

Ideas for Online Discussion

(Some ideas may apply to more than one standard of the NCTM Professional Standards for Teaching Mathematics.)

Standard 1: Worthwhile Mathematical Tasks

1. The concepts investigated in this lesson are fundamental to the study of algebra. Do you think the approach used was efficient? What content and what strategies are used that would also be appropriate for an Algebra I level class?
2. In this lesson, the concrete, hands-on portion followed the theoretical discussion of the algebraic concepts. Discuss the advantages and disadvantages of presenting the lesson in this order. Discuss the advantages and disadvantages of the reverse order.
3. What real-world problems dealing with rational or polynomial functions have you used with your classes?

Standard 2: The Teacher’s Role in Discourse

4. How did the teacher in this lesson orchestrate discourse both within the groups and for the class discussions? What did he do to monitor students and encourage student participation? What do you do in your classroom that is effective for you and your students?
5. How did the video teacher deal with incorrect or incomplete responses? How do you handle this in your classroom?

Standard 4: Tools for Enhancing Discourse

6. How was the graphing calculator used to enhance discourse within the groups and for the entire class discussion? With respect to the concepts covered in this lesson, how has the graphing calculator changed what you teach and at what level?

Standard 6: Analysis of Teaching and Learning

7. How did this video teacher analyze teaching and learning? What kinds of questions do you ask yourself as you reflect on your teaching?

Polynomial Functions

Group Members: _____

(Your signature indicates that you participated in preparing and agree with these responses.)

Please tell me the x-intercepts of these two equations.

1. $y = x^3 + x^2 - 12x$

2. $y = x^3 - 4x^2 - 15x + 18$

Please factor the two expressions below.

1. $x^2 + 4x - 21$

2. $x^4 - 11x^3 + 24x^2 + 44x - 112$

Please give the roots of the equations below.

1. $x^3 - 10x^2 + 17x + 28 = 0$

2. $x^4 - x^3 - 54x^2 + 144x = 0$

How are the roots, factors, and intercepts of an equation related?

Rational Functions

Group Members: _____
(Your signature indicates that you participated in preparing and agree with these responses.)

Please investigate the functions below. Discuss each graph. List as many interesting features as you can, and please make whatever observations you think may be useful. Discuss and share individual ideas before recording group ideas on this assessment sheet.

1. $y = \frac{x}{x-1}$

2. $y = \frac{x}{x^2 - x - 6}$

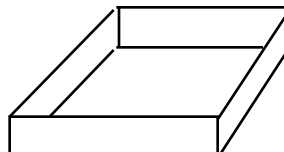
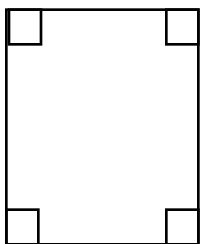
3. $y = \frac{x^2 + 2x - 15}{x}$

4. $y = \frac{x+2}{x^2 + x - 30}$

5. $y = \frac{x^2 - 4x - 5}{x - 5}$

The Box Problem

Problem Statement: You are the manager of a packaging company responsible for manufacturing identical rectangular boxes from rectangular sheets of cardboard, each sheet having the same dimensions. To save money, you want to manufacture boxes that will have the maximum possible volume. You need to determine what size squares (all the same size) to cut out of each corner of the rectangular sheets to form a box (without a top) that will have the maximum volume.



Procedure: Your group has been given several sheets of graph paper, scissors, and tape.

1. Have each group member construct a different size rectangular box by cutting four equal squares from each corner of a piece of graph paper, with each group member choosing a different size square. Fold up the loose ends to form a box, and tape the end of the sides together.
2. Measure the three dimensions of each box and use the measurements to compute the volume of each open box. Record the length of the square corner cutout and the corresponding volume in the table below.

Length of Side of Square Cutout	Volume of the Box

Polynomial Functions Selected Answers

X-Intercepts

1. x -intercepts: $-4, 0, 3$
2. x -intercepts: $-3, 1, 6$

Factors

1. $(x + 7)(x - 3)$
2. $(x + 2)(x - 2)(x - 4)(x - 7)$

Roots

1. $x = -1, 4, 7$
2. $x = -8, 0, 3, 6$

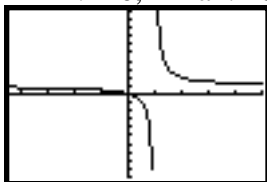
Conclusion: *There is a connection between the x -intercepts, factors, and roots. The x -intercepts and the roots are the same. If r_1 and r_2 are roots, then $(x - r_1)$ and $(x - r_2)$ are factors.*

Rational Functions Selected Answers

1. $y = \frac{x}{x-1}$

Xmin: -4.4, Xmax 5

Ymin: -10, Ymax: 10



Xmin: -5, Xmax 5

Ymin: -10, Ymax: 10

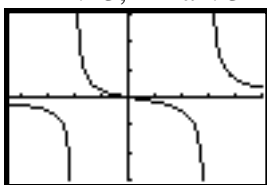


$x = 1$ is an asymptote. Selecting a good window is important with rational functions. The first graph has a better window than the second. See Mathematically Speaking for a discussion of this.

2. $y = \frac{x}{x^2 - x - 6}$

Xmin: -4.4, Xmax 5

Ymin: -3, Ymax: 3

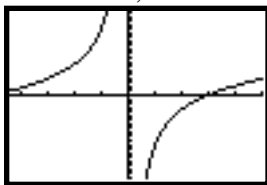


$x = -2$, and $x = 3$ are both asymptotes, and $(x + 2)$ and $(x - 3)$ are factors of the denominator.

3. $y = \frac{x^2 + 2x - 15}{x}$

Xmin: -4.4, Xmax 5

Ymin: -10, Ymax: 10

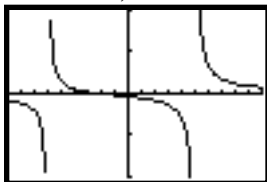


$x = 0$ is an asymptote, and x is the only factor of the denominator.

4. $y = \frac{x+2}{x^2+x-30}$

Xmin: -8.8, Xmax 10

Ymin: -2, Ymax: 2

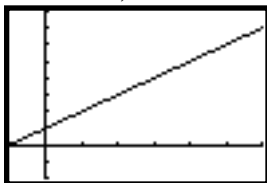


x = -5, and x = 5 are both asymptotes, and (x - 6) and (x - 5) are factors of the denominator.

5. $y = \frac{x^2 - 4x - 5}{x - 5}$

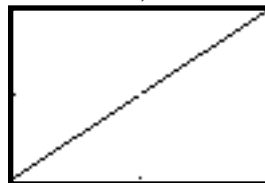
Xmin: -1, Xmax 6

Ymin: -2, Ymax: 8



Xmin: 4.5, Xmax 5.5

Ymin: 5.5, Ymax: 6.5

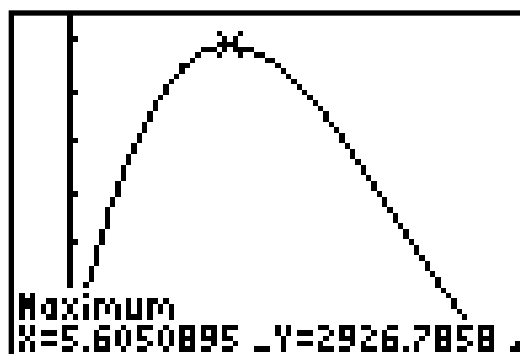


This particular function reduces to $y = x + 1$. The graph is a straight line with a hole in it. The value $x = 5$ is not part of the domain of this function because division by zero is not defined. Notice that in the graph on the right you can actually see the hole in the graph.

The Box Problem

Selected Answers

Length of Side of Square Cutout (Height)	Length	Width	Volume of the Box
1	37	28	1036
2	35	26	1820
3	33	24	2376
4	31	22	2728
5	29	20	2900
6	27	18	2916
7	25	16	2800
8	23	14	2576
9	21	12	2268
10	19	10	1900
11	17	8	1496
12	15	6	1080
13	13	4	676
14	11	2	308



The dimensions of the box with maximum volume (to the nearest tenth of a unit) are: height = 5.6, length = 27.8, and width = 18.8.